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The Spline Interpolation of Sequences Satisfying a Linear Recurrence Relation

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The properties of cardinal splines satisfying a linear recurrence relation and interpolating given data are studied. A necessary and sufficient condition is obtained for the existence of a unique cardinal spline of given degree fulfilling these requirements. The limiting behavior of such a family of splines as the degree tends to infinity is determined.

1. STATEMENT OF THE PROBLEM

In several recent papers, Schoenberg [6–8] has studied cardinal splines of degree n that satisfy

$$S(x+1) = tS(x) \tag{1.1}$$

for some fixed t and for all real x, and also interpolate at the integers the exponential function t^x. By a cardinal spline of degree n we mean a piecewise polynomial function $S(x)$ of degree n and continuity class C^{n-1}, defined on the real line and such that discontinuities in $S^{(n)}(x)$ occur only at the integers.

Schoenberg has shown [7, 8] that there is a unique cardinal spline of degree n, called the *exponential Euler spline*, satisfying these two conditions

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so long as t is not a zero of the Euler–Frobenius polynomial [6, 8] of degree n. If t is equal to one of these zeros (called *eigenvalues* for the degree n), there is no cardinal spline satisfying (1.1) and interpolating t^x at the integers. However, there do exist *only* for these values of t the *eigensplines*, which satisfy (1.1) and also the condition

$$S(\nu) = 0 \qquad (\nu = 0, \pm 1, \pm 2, \ldots).$$

In this paper we consider a generalization, which will now be described, of the problem solved by Schoenberg. Let $\{y_\nu\}_{-\infty}^{\infty}$ be a sequence of reals such that

$$\sum_{j=0}^{k} a_j y_{j+\nu} = 0 \tag{1.2}$$

for all integers ν, where the coefficients a_j are fixed real numbers, and $a_0 a_k \neq 0$. Let h be a fixed real number in the half-open interval $[0, 1)$. We then propose the following

PROBLEM. To construct a cardinal spline $S(x)$ of degree n that interpolates the data-sequence $\{y_\nu\}$ at the arguments $\nu + h$, i.e.,

$$S(\nu + h) = y_\nu \tag{1.3}$$

for all integers ν, and also satisfies the functional equation

$$\sum_{j=0}^{k} a_j S(x + j) = 0 \tag{1.4}$$

for all real x.

We shall show in Section 2 that the solution of the problem depends on the zeros of two polynomials. One is the characteristic polynomial of the recurrence relation (1.4), which is

$$P(u) = \sum_{j=0}^{k} a_j u^j \qquad (a_0 a_k \neq 0). \tag{1.5}$$

The other is

$$R_{n,h}(u) = \sum_{j=0}^{n} Q_{n+1}(j + h)\, u^{n-j}, \tag{1.6}$$

where $Q_{n+1}(x)$ is the "forward B-spline" of degree n (see [7, 8]). For $h = 0$ this reduces to $(1/n!)\,\Pi_n(u)$, where $\Pi_n(u)$ is the Euler–Frobenius polynomial of degree $n - 1$. We shall find that the problem stated has a unique solution if and only if $P(u)$ and $R_{n,h}(u)$ have no common zero. Clearly this is a plausible generalization of Schoenberg's result.

For $h = 0$, we have interpolation at the knots. For $P(u) = u - \lambda$, $h = 0$ or $\frac{1}{2}$, the problem reduces to the cases studied by Schoenberg [7, 8].

Let $\mathscr{S}_{n,P}$ denote the class of cardinal splines of degree n satisfying (1.4). We shall show in Section 3 that when $P(1) \neq 0$, there exists a cardinal spline $S^*(x) \in \mathscr{S}_{n,P}$ that is represented in $[0, k]$ by a single polynomial (i.e., the expected knots at $1, 2,..., k - 1$ are absent). In Section 4 this special cardinal spline $S^*(x)$ is shown to lead to a basis for the class $\mathscr{S}_{n,P}$.

Of particular interest is the case of $P(u) = (u - t)^k$ and $h = 0$, which leads to the exponential Euler splines of higher order treated in Section 5. In Section 6 we study the convergence when $n \to \infty$ of the interpolatory splines in $\mathscr{S}_{n,P}$ satisfying (1.3) with $h = 0$.

2. The Main Result

The forward B-splines $Q_{n+1}(x)$ have been thoroughly studied in [6, 7]. Every cardinal spline $S(x)$ of degree n is known [4] to have a unique representation of the form

$$S(x) = \sum_{\nu=-\infty}^{\infty} c_\nu Q_{n+1}(x - \nu). \tag{2.1}$$

We now formulate

Lemma 1. *A cardinal spline $S(x)$ of the form* (2.1) *belongs to $\mathscr{S}_{n,P}$ if and only if*

$$\sum_{j=0}^{k} a_j c_{\nu+j} = 0 \tag{2.2}$$

for every integer ν.

Proof. We have

$$\sum_{j=0}^{k} a_j S(x + j) = \sum_{j=0}^{k} a_j \sum_{\nu=-\infty}^{\infty} c_\nu Q_{n+1}(x + j - \nu)$$

$$= \sum_{\nu=-\infty}^{\infty} Q_{n+1}(x - \nu) \sum_{j=0}^{k} a_j c_{\nu+j}.$$

The last expression vanishes for all real x if and only if (2.2) holds for all integers ν. ∎

Using the relations

$$Q_{n+1}(x) = (1/n!)\, \Delta^{n+1}(x - n - 1)_+^n$$

and

$$(x - n - 1)_+^n = (x - n - 1)(x - n - 1)_+^{n-1} \qquad (n = 1, 2, \ldots), \quad (2.3)$$

where

$$x_+ = \max(0, x) = \tfrac{1}{2}(x + | x |),$$

and applying to (2.3) the formula for the $(n + 1)$th finite difference of a product, we obtain, after some simplification, the recurrence relation

$$Q_{n+1}(x) = (1/n)[xQ_n(x) + (n + 1 - x)\, Q_n(x - 1)], \quad (2.4)$$

valid for all real x. This is also a particular case of the recurrence relation for B-splines with general knots due to de Boor, Mansfield, and Cox [1, 2]. From (1.6) and (2.4) we deduce the recurrence relation

$$R_{n,h}(x) - xR_{n-1,h}(x) = \frac{1 - x}{n} [(1 - h)\, R_{n-1,h}(x) + xR'_{n-1,h}(x)], \quad (2.5)$$

which will be used to prove the following lemma.

LEMMA 2. *For $n = 1, 2, \ldots$, the zeros of $R_{n,h}(u)$ are simple and negative.*

Proof. For $h = 0$,

$$R_{n,0}(u) = (1/n!)\, \Pi_n(u),$$

and the zeros of $\Pi_n(u)$ are known to be simple and negative [7, 8].

For $h > 0$, we follow Schoenberg, using induction on n, and suppose that the zeros of $R_{n-1,h}(u)$ are

$$\lambda_{n-1} < \lambda_{n-2} < \cdots < \lambda_2 < \lambda_1 < 0.$$

Then, it follows from (2.5) that

$$R_{n,h}(\lambda_j) = (1/n)\, \lambda_j(1 - \lambda_j) R'_{n-1,h}(\lambda_j) \qquad (j = 1, 2, \ldots, n - 1).$$

Since $R_{n,h}(0) = Q_{n+1}(1 - h) > 0$, it follows that $(-1)^{j-1} R'_{n-1,h}(\lambda_j) > 0$ for all j, and therefore

$$(-1)^j R_{n,h}(\lambda_j) > 0 \qquad (j = 1, 2, \ldots, n - 1).$$

Since the coefficient of u^n in $R_{n,h}(u)$ is $Q_{n+1}(h) > 0$, it follows that $R_{n,h}(u)$ has a zero in each of the n intervals

$$(-\infty, \lambda_{n-1}), (\lambda_{n-1}, \lambda_{n-2}), \ldots, (\lambda_1, 0).$$

Since

$$R_{1,h}(u) = hu + 1 - h,$$

the induction is complete. ∎

In the proof of the following theorem it will be convenient to use the so-called displacement operator E of the calculus of finite differences defined by

$$Ef(x) = f(x+1)$$

for every function f. Then (1.4) can be written as

$$P(E)\,S(x) = 0. \tag{2.6}$$

THEOREM 1. *Given a real polynomial $P(u)$ of the form* (1.5) *and a sequence* $\{y_\nu\}$ *satisfying* (1.2), *there is a unique cardinal spline $S(x) \in \mathscr{S}_{n,P}$ satisfying* (1.3) *if and only if $P(u)$ and $R_{n,h}(u)$ have no common zero.*

Proof. If such a cardinal spline $S(x)$ exists, it has a unique representation of the form (2.1) with coefficients c_ν that satisfy (2.2), by Lemma 1. We then have

$$\sum_{\nu=-\infty}^{\infty} c_\nu Q_{n+1}(j + h - \nu) = y_j \tag{2.7}$$

for all integers j.

Conversely, if there exists a sequence $\{c_\nu\}$ satisfying (2.2) and (2.7), then (2.1) exhibits a spline $S(x)$ having the required properties.

Consider now the more limited system of equations

$$\begin{aligned} &\sum_{\nu=-\infty}^{\infty} c_\nu Q_{n+1}(j+h-\nu) = y_j \qquad (j = 0, 1,..., k-1) \\ &\sum_{\nu=0}^{k} a_\nu c_{\nu-j} = 0 \qquad (j = 1, 2,..., n). \end{aligned} \tag{2.8}$$

In view of the limited support of $Q_{n+1}(x)$, (2.8) is a linear system of $n + k$ equations in the $n + k$ unknowns, $c_{-n}, c_{-n+1},..., c_{k-1}$. Moreover, it is equivalent to the more extensive system consisting of (2.2) and (2.7), since, because $a_0 a_k \neq 0$, the recurrence relations (1.2) and (2.2) can be used to extend the equations uniquely to all j.

Thus, a spline $S(x)$ having the required properties exists if and only if the square matrix of coefficients of (2.8) is nonsingular. This is the case if and only if the corresponding homogeneous system has only the trivial solution in which all the unknowns vanish.

Now, the existence of a solution of the homogeneous system is tantamount to the existence of a sequence $\{\tilde{c}_\nu\}$ satisfying both the difference equations $P(E)\,\tilde{c}_\nu = 0$ and $R_{n,h}(E)\,\tilde{c}_\nu = 0$ for all integers ν. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the zeros of $R_{n,h}(u)$. Since they are distinct by Lemma 2, a solution $\tilde{c}_\nu$ of $R_{n,h}(E)\,\tilde{c}_\nu = 0$ must be of the form

$$\tilde{c}_\nu = \sum_{j=1}^{n} A_j \mu_j^{\nu}.$$

Consequently,

$$P(E)\,\tilde{c}_\nu = \sum_{j=1}^{n} A_j P(\mu_j)\,\mu^\nu = 0$$

for all integers ν. Since the μ_j are distinct, this implies

$$A_j P(\mu_j) = 0 \qquad (j = 1, 2, \ldots, n).$$

Now, if $P(u)$ and $R_{n,h}(u)$ have no common zero, $P(\mu_j) \neq 0$ for all j and so $A_j = 0$ for all j. In other words, $\tilde{c}_\nu$ is the trivial solution.

On the other hand, if $P(u)$ and $R_{n,h}(u)$ have a common zero, then for some j, say $j = d$, $P(\mu_d) = 0$. Thus,

$$\tilde{c}_\nu = \mu_d^{\nu}$$

is a nontrivial solution of the homogeneous system. ∎

EXAMPLE 1. Consider the Fibonacci sequence

ν	-4	-3	-2	-1	0	1	2	3	4	5	6
f_ν	-3	2	-1	1	0	1	1	2	3	5	8

satisfying the recurrence relation

$$f_{j+2} - f_{j+1} - f_j = 0$$

for all j. The characteristic polynomial

$$P(u) = u^2 - u - 1$$

has the zeros

$$\lambda_1 = \frac{1 + 5^{1/2}}{2}, \qquad \lambda_2 = \frac{1 - 5^{1/2}}{2}.$$

We apply Theorem 1 to construct the interpolating spline function $S_n(x)$ satisfying

$$S_n(x+2) - S_n(x+1) - S_n(x) = 0$$

for all real x. For this we have to show that $P(u)$ and $\Pi_n(u)$ have no common zero. This we see as follows: $P(u)$ and $\Pi_n(u)$ are polynomials with rational coefficients and $P(u)$ is irreducible over the rational field. If they had a common zero, then $P(u)$ would have to be a divisor of $\Pi_n(u)$, which is impossible because it would follow that $\Pi_n(u)$ has the positive zero λ_1.

Hence, by Theorem 1, for every natural number n there exists a unique cardinal spline function $S_n(x)$ of degree n, such that

$$S_n(\nu) = f_\nu$$

for all integers ν. We call $S_n(x)$ the *Fibonacci spline* of degree n.

$S_n(x)$ is uniquely defined in [0, 2] by the $n+2$ conditions

$$S_n(0) = 0, \quad S_n(1) = 1, \quad S_n(2) = 1,$$

$$S_n^{(r)}(2) = S_n^{(r)}(0) + S_n^{(r)}(1) \qquad (r = 1, 2,..., n-1).$$

Solving these elementary problems for $n = 2$ and $n = 3$, we find that

$$S_2(x) = 4x - 3x^2 + 5(x-1)_+^2 \qquad (0 \leqslant x \leqslant 2),$$

$$S_3(x) = (1/11)[-6x + 36x^2 - 19x^3 + 31(x-1)_+^3] \qquad (0 \leqslant x \leqslant 2$$

3. The Polynomials $A_n(x; P)$

We begin by proving

Lemma 3. *Let $P(x)$ be given by* (1.5) *with $P(1) \neq 0$. Then, for every integer $n \geqslant 0$, there is a unique monic polynomial $A_n(x; P)$ of degree n such that*

$$P(E)\, A_n(x; P) = P(1)\, x^n. \tag{3.1}$$

The polynomials $A_n(x; P)$ are given by the generating function

$$\sum_{n=0}^{\infty} A_n(x; P) \frac{z^n}{n!} = \frac{e^{xz} P(1)}{P(e^z)}. \tag{3.2}$$

Proof. Let the polynomials $A_n(x; P)$ be defined by (3.2). Since $P(1) \neq 0$. $P(1)/P(e^z)$ has a formal expansion of the form

$$\frac{P(1)}{P(e^z)} = 1 + b_1 z + b_2 z^2 + \cdots.$$

It follows that $A_n(x; P)$ is in fact a monic polynomial of degree n. Moreover, since

$$P(E)\, e^{xz} = P(e^z)\, e^{xz},$$

it follows from (3.2) that (3.1) holds for each n. This proves the existence of polynomials having the required properties.

If there are two different monic polynomials of degree n, $A_n(x; P)$ and $q(x)$, satisfying (3.1), then

$$P(E)(A_n(x; P) - q(x)) = 0,$$

which is impossible, since

$$P(E)\, x^\nu = P(1)\, x^\nu + \cdots. \quad \blacksquare$$

Remark 1. It follows easily from (3.1) or (3.2) that the polynomials $\{A_n(x; P)\}$ form an Appell set, i.e.,

$$A_n'(x; P) = nA_{n-1}(x; P), \qquad A_0(x; P) = 1.$$

Relation (3.2) leads to a recursive method for obtaining $A_n(x; P)$ explicitly. In fact, multiplying both sides of (3.2) by $P(e^z)$, differentiating with respect to z, and setting $z = 0$ gives

$$\sum_{j=0}^{n} \binom{n}{j} A_{n-j}(x; P)[P(E)\, x^j]_{x=0} = P(1)\, x^n. \tag{3.3}$$

Remark 2. If $P(1) = P'(1) = \cdots = P^{(m-1)}(1) = 0$, $P^{(m)}(1) \neq 0$, then we define the polynomials by the relations

$$P(E)\, A_n(x; P) = n(n-1) \cdots (n-m+1)\, x^{n-m} P^{(m)}(1), \tag{3.1a}$$

or by the generating function

$$\sum_{n=0}^{\infty} A_n(x; P) \frac{z^n}{n!} = \frac{z^m e^{xz} P^{(m)}(1)}{P(e^z)}. \tag{3.2a}$$

EXAMPLE 2. If

$$P(u) = \frac{u-a}{1-a}, \qquad a \neq 1,$$

then

$$[P(E)\, x^j]_{x=0} = 1 \qquad (j = 0)$$
$$= (1-a)^{-1} \qquad (j = 1, 2, \ldots),$$

and (3.3) reduces to

$$A_n(x; P) = x^n + \frac{1}{a-1} \sum_{j=1}^{n} \binom{n}{j} A_j(x; P). \tag{3.4}$$

On the other hand,

$$A_n(x; P) = [P(E)]^{-1}\, x^n = \frac{1-a}{E-a}\, x^n = \left(1 - \frac{\Delta}{a-1}\right)^{-1} x^n$$

gives

$$A_n(x; P) = \sum_{\nu=0}^{n} (a-1)^{-\nu} \Delta^{\nu} x^n \qquad (n = 0, 1, \ldots). \tag{3.5}$$

It is easily verified that (3.5) satisfies (3.4).

EXAMPLE 3. Taking $P(u) = u - 1$ in (3.2a), we have the well known relation for Bernoulli polynomials,

$$\sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{ze^{zx}}{e^z - 1}.$$

The relation between the polynomials $A_n(x; P)$ and the cardinal splines of $\mathscr{S}_{n,P}$ is brought out by

THEOREM 2. *Given a polynomial $P(u)$ of the form* (1.5) *with* $P(1) = 1$, *there exists a unique cardinal spline $S(x) \in \mathscr{S}_{n,P}$ that coincides on* $[0, k]$ *with $A_n(x; P)$ of Lemma* 3.

Conversely, if $S(x) \in \mathscr{S}_{n,P}$ and is represented on $[0, k]$ *by a single polynomial $q_n(x)$ (i.e., the expected knots at* $1, 2, \ldots, k-1$ *are absent), then* $q_n(x) = cA_n(x; P)$ *for some constant c.*

Proof. Since $a_0 a_k \neq 0$, the restriction to $[0, k]$ of the polynomial $A_n(x; P)$ has a unique extension to the entire real line by means of the recurrence relation (1.4). To prove the first part of the theorem, it is sufficient to show that this unique extension belongs to C^{n-1}, and therefore to $\mathscr{S}_{n,P}$. As a first step, we show that the knot at $x = k$ is simple.

Indeed, if $S(x)$ denotes the unique extension, it follows from (1.4) and (3.1) that for $x \in [0, 1]$,

$$S(x + k) = A_n(x + k; P) - a_k^{-1} x^n,$$

since $P(1) = 1$. This shows that the extension to $(0, k + 1)$ belongs to C^{n-1}.

Now, it follows from (1.4) that

$$\sum_{j=0}^{k} a_j(\text{jump of } S^{(r)}(x) \text{ at } x = \nu + j) = 0$$

for every integer ν and every nonnegative integer r. Therefore, since $a_0 a_k \neq 0$, and the jumps of the first $n - 1$ derivatives of $S(x)$ at $x = 1, 2,..., k$ vanish, it can be shown by induction on ν that $S(x)$ and its first $n - 1$ derivatives are continuous at the integers.

On the other hand, let $S(x) \in \mathscr{S}_{n,P}$ and let $S(x) = q(x)$ in $[0, k]$, where $q(x) \in \pi_n$. We use π_n to denote the class of polynomials of degree n or less. For $x \in [k, k + 1]$, we have

$$S(x) = q(x) + c(x - k)_-^n$$

for some real c. Then, (1.4) implies

$$P(E)\, q(x) = -ca_k x^n \qquad (x \in [0, 1]), \tag{3.6}$$

from which it follows that (3.6) holds for all real x. By Lemma 3,

$$q(x) = c_1 A_n(x; P),$$

where c_1 is some constant. ∎

4. A Basis for $\mathscr{S}_{n,P}$

Let $S_{n,P}(x)$ denote the cardinal spline of the class $\mathscr{S}_{n,P}$ that is represented on $[0, k]$ by the polynomial $A_n(x; P)$. This unique spline has a useful property which we formulate in

Theorem 3. *With $P(u)$ given by (1.5) with $P(1) \neq 0$, there do not exist real constants $b_0, b_1, ..., b_l$, $b_0 b_l \neq 0$, $l < k$, such that*

$$\sum_{j=0}^{l} b_j S_{n,P}(x + j) = 0 \tag{4.1}$$

for all real x.

Consequently, the k splines,

$$S_{n,P}(x), S_{n,P}(x + 1),..., S_{n,P}(x + k - 1), \tag{4.2}$$

constitute a basis for $\mathscr{S}_{n,P}$.

In other words, this theorem says that $S_{n,P}(x)$ does not satisfy a linear recurrence relation of order less than k, and the k translates (4.2) form a basis for $\mathscr{S}_{n,P}$.

Proof. Let l be the smallest integer for which a relation of the form (4.1) holds, and let

$$B(u) = \sum_{j=0}^{l} b_j u^j.$$

Then, it follows from Lemma 1 and from the theory of linear difference equations with constant coefficients that the set of zeros of $B(u)$ is a subset of the set of zeros of $P(u)$. But (4.1) implies, in particular,

$$\sum_{j=0}^{l} b_j A_n(x + j; P) = 0 \qquad (x \in [0, 1]), \tag{4.3}$$

and so (4.3) is an identity for all x. Thus, the coefficient of x^n in (4.3) must vanish, i.e.,

$$\sum_{j=0}^{l} b_j = B(1) = 0.$$

That is, 1 is a zero of $B(u)$ and hence of $P(u)$. This contradicts the hypothesis that $P(1) \neq 0$.

By Lemma 1, a basis for $\mathscr{S}_{n,P}$ has exactly k elements. Since there is no linear recurrence of the form (4.1) with $l < k$, the k splines (4.2) are linearly independent and hence form a basis for $\mathscr{S}_{n,P}$. ∎

EXAMPLE 4. If $P(u) = u^2 - u - 1$,

$$S_{3,P}(x) = x^3 + 3x^2 + 15x + 31 \qquad (0 \leqslant x \leqslant 2), \tag{4.4}$$

while $S_{3,P}(x)$ is extended to the remainder of the real line by means of the recurrence

$$S_{3,P}(x + 2) - S_{3,P}(x + 1) - S_{3,P}(x) = 0,$$

which holds for all real x. Replacing x by $x + 1$ in (4.4), we find that in [0, 1],

$$S_{3,P}(x + 1) = x^3 + 6x^2 + 24x + 50.$$

Therefore in [0, 1],

$$-\frac{50}{11} S_{3,P}(x) + \frac{31}{11} S_{3,P}(x + 1) = \frac{1}{11}(-19x^3 + 36x^2 - 6x),$$

which agrees with the expression given in Example 1 for the Fibonacci spline of degree 3. In fact, for all real x, the Fibonacci spline $S_3(x)$ is given by

$$S_3(x) = -\frac{50}{11} S_{3,P}(x) + \frac{31}{11} S_{3,P}(x+1).$$

5. Exponential Euler Splines of Higher Order

In [7] Schoenberg has studied in great detail the cardinal splines $S_{n,P}(x)$ in the particular case in which $P(E)$ is of the form $E - t$. These are called exponential Euler splines and have many interesting properties.

A natural generalization is the case in which

$$P(E) = (E - t)^{r+1}, \tag{5.1}$$

where t is a constant (not 0 or 1) and r is a given positive integer. We shall call a cardinal spline satisfying (2.6) with $P(E)$ given by (5.1) an *exponential Euler spline of order* r. We shall call t the base of the exponential Euler spline. Such a spline interpolates at the integers data of the form $\{p(\nu)\, t^\nu\}$, where $p(\nu)$ is some polynomial in ν of degree r. (This use of the term "order" differs from that of some writers, who define the order of a spline as one more than the degree.)

The polynomials associated with these splines by means of Lemma 3 will be called *exponential Euler polynomials of order* r. For $t = -1$ (see [5]), they are the standard Euler polynomials of higher order. We shall denote by $A_{n,r}(x; t)$ the polynomial $A_n(x; P)$ in the case when $P(E)$ is given by (5.1).

It follows from Lemma 3 (and especially from the uniqueness of the polynomials $A_n(x; P)$) that

$$(E - t)^{r+1} A_{n,r}(x; t) = (1 - t)^{r+1} x^n \qquad (n = 1, 2,\dots) \tag{5.2}$$

and

$$(E - t)\, A_{n,r}(x; t) = (1 - t)\, A_{n,r-1}(x; t) \qquad (r = 1, 2,\dots). \tag{5.3}$$

The following lemma will be utilized in the convergence proof of the next section.

Lemma 4. *The polynomials $A_{n,r}(x; t)$ satisfy the recurrence relation*

$$(r + 1)\, A_{n,r+1}(x; t) = ((t - 1)/t)[A_{n+1,r}(x; t) + (r - x + 1)\, A_{n,r}(x; t)] \qquad (r = 0, 1,\dots). \tag{5.4}$$

Proof. The proof will be by induction on r. For $r = 0$, (5.4) becomes

$$A_{n,1}(x; t) = ((t - 1)/t)[A_{n+1,0}(x; t) + (1 - x)\, A_{n,0}(x; t)]. \tag{5.5}$$

Both members of (5.5) are polynomials in x of degree n. It will therefore be established if we can show that operating on both members with $E - t$ yields an identity. An easy computation using (5.2) and (5.3) shows that this is the case.

Now, suppose (5.4) holds for $r = l$, and consider the corresponding relation for $r = l + 1$. Again, both members are polynomials in x of degree n, and the relation is established if operating with $E - t$ yields an identity. In fact, operating on the right member and using (5.3) gives

$$((t-1)/t)\{(1-t)[A_{n+1,l}(x;t) + (l - x + 1)\,A_{n,l}(x;t)] - tA_{n,l+1}(x;t)\}.$$

Under the induction hypothesis, this reduces to

$$\begin{aligned}(1-t)[(l-1)\,A_{n,l+1}(x;t) + A_{n,l+1}(x;t)]\\ = (E-t)[(l+2)\,A_{n,l+2}(x;t)],\end{aligned}$$

and the induction is complete.

It is not difficult to show that the polynomials $A_{n,r}(x;t)$ have the further property that

$$A_{n,r}(-x;t) = (-1)^n\,A_{n,r}(x + r + 1; t^{-1}).$$

It follows from Lemma 1 that a given exponential Euler spline $S(x)$ of degree n and order r has a unique representation of the form

$$S(x) = \sum_{\nu=-\infty}^{\infty} q(\nu)\,t^{\nu} Q_{n+1}(x - \nu), \tag{5.6}$$

where $q(\nu)$ is a polynomial of (strict) degree r in ν. One may ask what is the relationship between $q(\nu)$ and $p(\nu)$, where $p(\nu)\,t^{\nu}$ is the function interpolated by $S(x)$ at the integers ν. In order to elucidate this relationship, we shall need a suitable basis for the space of exponential Euler splines of degree n and order r. To this end, let $S^*_{n,r}(x;t)$ denote the unique exponential Euler spline of degree n and order r that interpolates $\{\binom{\nu}{r}\,t^{\nu-r}\}$.

Evidently $S^*_{n,0}(x;t)$ is merely the exponential Euler spline $S_n(x;t)$ of [7, 8]. It is also clear that $S^*_{n,0}(x;t), S^*_{n,1}(x;t),\ldots, S^*_{n,r}(x;t)$ are a basis for the space of exponential Euler splines of degree n and order r. Moreover, it is easily verified that

$$(E-t)\,S^*_{n,r}(x;t) = S^*_{n,r-1}(x;t). \tag{5.7}$$

We wish to allow for the possibility that t may be complex. When this is the case, an exponential Euler spline of degree n and order r is a complex-valued function of a real variable. The latter fact, however, does not materially change the properties of this cardinal spline. It is, as always, uniquely determined by the (complex) values $\{p(\nu)\,t^\nu\}$ that it interpolates at the integers.

From the definition of the Euler-Frobenius polynomials, it is easily deduced that

$$S^*_{n,0}(x;t) = S_n(x;t) = \frac{n!}{\Pi_n(t)} \sum_{\nu=-\infty}^{\infty} t^{\nu-n} Q_{n+1}(x-\nu). \tag{5.8}$$

For a fixed x, the last expression in (5.8) may be regarded as an analytc function of the complex variable t. Its poles are at the origin and the zeros of $\Pi_n(t)$, all of which are known to be negative. Therefore, $S_n(x;t)$ is infinitely differentiable with respect to t on the entire complex plane with the origin and the negative real axis excluded. We shall need the following lemma.

LEMMA 5. *Let the complex quantity t be neither zero nor negative. Then*

$$S^*_{n,r}(x;t) = \frac{1}{r!}\frac{\partial^r}{\partial t^r} S_n(x;t). \tag{5.9}$$

Proof. It follows from (5.8) that the right member of (5.9) is of the form

$$\sum_{\nu=-\infty}^{\infty} p_{n,r}(\nu)\, t^{\nu-r} Q_{n+1}(x-\nu), \tag{5.10}$$

where $p_{n,r}(\nu)$ is a polynomial of degree r in ν (having functions of t as coefficients). This expression is of the form (5.6) and is therefore an exponential Euler spline of degree n and order r. Its value for $x = \nu$, an integer, is given by

$$\frac{1}{r!}\frac{d^r}{dt^r}(t^\nu) = \binom{\nu}{r} t^{\nu-r}.$$

Therefore, (5.9) follows from the uniqueness property of Theorem 1. ∎

An arbitrary polynomial $p(\nu)$ of degree r can be expressed in the form

$$p(\nu) = \sum_{l=0}^{r} \binom{\nu}{l} \Delta^l p(0),$$

and therefore the exponential Euler spline $S(x)$ of degree n and order r that interpolates $\{p(\nu)\,t^\nu\}$ can be expressed in the form

$$S(x) = \sum_{l=0}^{r} t^l \Delta^l p(0)\, S^*_{n,l}(x;t).$$

Substituting for $S_{n,l}^{*}(x; t)$ the expression obtained from (5.8) gives

$$S(x) = \sum_{l=0}^{r} \Delta^{l} p(0) \sum_{\nu=-\infty}^{\infty} p_{n,r}(\nu)\, t^{\nu} Q_{n+1}(x-\nu).$$

Consequently, the polynomial $q(\nu)$ of (5.6) is given by

$$q(\nu) = \sum_{l=0}^{r} p_{n,r}(\nu)\, \Delta^{l} p(0),$$

where

$$p_{n,r}(\nu) = \frac{n!}{r!}\, t^{r-\nu} \frac{d^r}{dt^r} \frac{t^{\nu+n}}{\Pi_n(t)}\,.$$

EXAMPLE 5. $S_{2,0}^{*}(x; t) = S_2(x; t)$ is given in [0, 1] by

$$S_{2,0}^{*}(x; t) = 1 + \frac{2(t-1)}{t+1}\, x + \frac{(t-1)^2}{t+1}\, x^2. \tag{5.11}$$

The expression in [0, 1] for $S_{2,1}^{*}(x; t)$ is

$$S_{2,1}^{*}(x; t) = \frac{4}{(t+1)^2}\, x + \frac{(t-1)(t+3)}{(t+1)^2}\, x^2,$$

obtained by differentiating (5.11) with respect to t.

6. THE LIMIT OF $S_{n,P}(x)$ AS $n \to \infty$

In the case in which there is, for each n, a unique $S_{n,P}(x)$ satisfying (1.4) and interpolating the data $\{y_\nu\}$, do the cardinal splines $S_{n,P}(x)$ approach a definite limiting function as $n \to \infty$? We shall see that the answer is affirmative when $P(x)$ has no negative zero.

Note that if $P(x)$ has no negative zero, then $S_{n,P}(x)$ is, in fact, uniquely determined by Theorem 2, because in such a case $P(x)$ has no zeros in common with any of the polynomials $\Pi_n(x)$, since the zeros of the latter polynomials are known to be all negative.

We shall consider first the case of (generalized) exponential Euler splines, in which $P(x)$ has only one (in general, multiple) zero. If a sequence $\{S_{n,P}(x)\}$ satisfies (5.1) and interpolates at the integers the data $\{q(\nu)\, t^{\nu}\}$, where $q(x) \in \pi_r$, then it is reasonable to conjecture that the limiting function is $q(x)\, t^{x}$. However, we must bear in mind that t is not restricted to real values, and if

$$t = |\, t\,|\, e^{i\theta},$$

t^x takes on all the values

$$|t|^x \exp(i\theta + 2\pi\mu i)$$

for all integers μ.

However, in the case of the simple exponential Euler splines ($r = 0$), Schoenberg [7] has shown that if t is not negative,

$$\lim_{n\to\infty} S_n(x; t) = |t|^x e^{i\alpha x},$$

where

$$t = |t| e^{i\alpha} \qquad (-\pi < \alpha < \pi). \tag{6.1}$$

In other words, the interpolating cardinal spline converges to the principal value.

In order to prove an analogous result for exponential Euler splines of order r, we shall need the following lemma.

LEMMA 6. *For all real x and $t \neq 1$,*

$$S_{n,1}^*(x + 1; t) = (x + 1) S_n(x; t) - \frac{\Pi_{n+1}(t)}{(n + 1)(t - 1)\Pi_n(t)} [S_{n+1}(x; t) - S_n(x; t)] \tag{6.2}$$

Proof. By (5.7),

$$(E - t) S_{n,1}^*(x + 1; t) = S_n(x + 1; t).$$

On the other hand,

$$(E - t) S_{n+1}(x; t) = 0 = (E - t) S_n(x; t),$$

and it is easily verified that

$$(E - t)[(x + 1) S_n(x; t)] = S_n(x + 1; t).$$

In other words, both members of (6.2) yield identical results when operated on with $E - t$. Accordingly, each member satisfies

$$F(x + 1) = tF(x) + S_n(x + 1; t).$$

It follows that (6.2) holds for all real x if it can be shown to hold in some interval of unit width. Consider the interval $-1 \leqslant x \leqslant 0$, and let $B_n(x)$ denote the polynomial of degree n that coincides with $S_{n,1}^*(x + 1; t)$ in $(-1, 0)$. By Theorem 3, $B_n(x)$ has a unique expression of the form

$$B_n(x) = b_1 A_{n,1}(x + 1; t) + b_2 A_{n,1}(x + 2; t). \tag{6.3}$$

By means of (5.5) and the substitution

$$A_{n,1}(x+2;t) = tA_{n,1}(x+1;t) + (1-t)\,A_n(x+1;t)$$

derived from (5.3), (6.3) can be rewritten in the form

$$B_n(x) = c_1 A_{n+1}(x+1;t) + (c_2 - c_1 x)\,A_n(x+1;t), \tag{6.4}$$

where c_1 and c_2 are uniquely determined by the relations

$$B_n(-1) = S^*_{n,1}(0) = 0, \qquad B_n(0) = S^*_{n,1}(1) = 1. \tag{6.5}$$

On the other hand, it follows from [7, (3.7) and (3.15)] that

$$\Pi_n(t) = (1/n!)(t-1)^{-n}\,A_n(0;t) \qquad (n = 1, 2,\ldots),$$

and therefore the right member of (6.2) becomes

$$(x+1)\,S_n(x;t) - \frac{A_{n+1}(0;t)}{A_n(0;t)}\,[S_{n+1}(x;t) - S_n(x;t)].$$

For $-1 \leqslant x \leqslant 0$, this is equal to

$$(x+1)\frac{A_n(x+1;t)}{A_n(0;t)} - \frac{A_{n+1}(0;t)}{A_n(0;t)}\left[\frac{A_{n+1}(x+1;t)}{tA_{n+1}(0;t)} - \frac{A_n(x+1;t)}{tA_n(0;t)}\right]. \tag{6.6}$$

Now, (6.6) is of the form of the right member of (6.4), and, moreover, it reduces to zero for $x = -1$ and to unity for $x = 0$. Therefore (6.6) is the polynomial of degree n uniquely determined by (6.4) and (6.5), and (6.2) is established. ∎

Let ρ denote the shortest distance from t to the negative real axis. In other words,

$$\begin{aligned} \rho &= |\,t\,| && (\operatorname{Re} t \geqslant 0)\\ &= |\operatorname{Im} t\,| && (\operatorname{Re} t < 0). \end{aligned}$$

We then have

LEMMA 7. *For $l = 0, 1,\ldots,$*

$$\frac{1}{l!}\left|\frac{d^l}{dt^l}\frac{\Pi_{n+1}(t)}{(n+1)(t-1)\,\Pi_n(t)}\right| \leqslant \frac{1}{|\,t-1\,|^{l+1}} + \frac{|\,t\,| + \rho}{\rho^{l+1}}.$$

Proof. Since $\Pi_n(t)$ satisfies [7] the recurrence relation,

$$\Pi_{n+1}(t) = (1+nt)\,\Pi_n(t) + t(1-t)\,\Pi_n{}'(t),$$

we have

$$\frac{\Pi_{n+1}(t)}{(n+1)(t-1)\,\Pi_n(t)} = \frac{n}{n+1} + \frac{1}{t-1} - \frac{t\Pi_n'(t)}{(n+1)\,\Pi_n(t)}. \tag{6.7}$$

We therefore consider the derivatives of $t\Pi_n'(t)/\Pi_n(t)$.

Let the zeros of $\Pi_n(t)$ be $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$. Then,

$$\frac{\Pi_n'(t)}{\Pi_n(t)} = \sum_{j=1}^{n-1} \frac{1}{t-\lambda_j}. \tag{6.8}$$

Consequently,

$$t\Pi_n'(t) \qquad \sum_{j=1}^{n-1} \frac{t-\lambda_j+\lambda_j}{t-\lambda_j} = n-1+\sum_{j=1}^{n-1}\frac{\lambda_j}{t-\lambda_j}.$$

and so

$$\frac{d^l}{dt^l}\frac{t\Pi_n'(t)}{\Pi_n(t)} = (-1)^l\, l!\sum_{j=1}^{n-1}\frac{\lambda_j}{(t-\lambda_j)^{l+1}} = (-1)^l\, l!\sum_{j=1}^{n-1}\frac{\lambda_j-t+t}{(t-\lambda_j)^{l+1}}$$

$$= (-1)^l\, l!\left[\sum_{j=1}^{n-1} t(t-\lambda_j)^{-l-1} - \sum_{j=1}^{n-1}(t-\lambda_j)^{-l}\right]$$

for $l = 1, 2, \ldots$. Since

$$\frac{1}{|t-\lambda_j|} \leqslant \frac{1}{\rho}$$

for all j,

$$\left|\frac{d^l}{dt^l}\frac{t\Pi_n'(t)}{\Pi_n(t)}\right| \leqslant (n-1)\, l!\,\frac{|t|+\rho}{\rho^{l+1}},$$

and substitution in (6.7) gives

$$\left|\frac{d^l}{dt^l}\frac{\Pi_{n+1}(t)}{(n+1)(t-1)\,\Pi_n(t)}\right| \leqslant \frac{l!}{|t-1|^{l+1}} + \frac{n-1}{n+1}\,\frac{l!(|t|+\rho)}{\rho^{l+1}}$$

$$\leqslant l![|t-1|^{-l-1} + (|t|+\rho)\,\rho^{-l-1}].$$

A simple calculation shows that this is true also for $l = 0$. ∎

We now prove

THEOREM 4. *If t is not negative and not equal to* 0 *or* 1, *then for every real x*,

$$\lim_{n\to\infty} S_{n,r}^*(x;t) = \binom{x}{r}\,|t|^{x-r}\,e^{i\alpha(x-r)}$$

where α is given by (6.1).

Proof. Schoenberg has shown (see [7, Part III]) that, for all real x,

$$| S_n(x; t) - | t |^x e^{i\alpha x} | < M | t |^x \gamma^n, \tag{6.9}$$

where M is a positive constant and $\gamma \in (0, 1)$ is given by

$$\gamma = \max \left(\left| \frac{t_0}{t_1} \right|, \left| \frac{t_0}{t_{-1}} \right| \right),$$

with

$$t_k = \log | t | + i\alpha + 2k\pi i.$$

We shall generalize this result by showing that for all real x and for $\nu = 0, 1, \ldots$,

$$| S_{n,\nu}^*(x; t) - \binom{x}{\nu} | t |^{x-\nu} e^{i\alpha(x-\nu)} | \leqslant M_\nu(x) | t |^{x-\nu} (n + 1)^\nu \gamma^n, \tag{6.10}$$

where $M_\nu(x)$ is a positive continuous function of x independent of n. The proof will be by induction on ν. For $\nu = 0$, (6.10) reduces to Schoenberg's result, taking $M_0(x) = M$. Suppose that (6.10) is true for $\nu = 0, 1, \ldots, r$.

From (6.2) we obtain by r-fold differentiation with respect to t and division by $(r + 1)!$,

$$\begin{aligned} S_{n,r+1}^*(x; t) &= \frac{x}{r+1} S_{n,r}^*(x - 1; t) \\ &\quad - \frac{1}{r+1} \sum_{l=0}^{r} \frac{1}{l!} \frac{d^l}{dt^l} \left[\frac{\Pi_{n+1}(t)}{(n+1)(t-1)\, \Pi_n(t)} \right] \\ &\quad \times [S_{n+1,r-l}^*(x - 1; t) - S_{n,r-l}^*(x - 1; t)]. \end{aligned}$$

Subtracting $\binom{x}{r+1} | t |^{x-r-1} e^{i\alpha(x-r-1)}$ from both sides and applying Lemma 7, we have

$$\begin{aligned} &\left| S_{n,r+1}^*(x; t) - \binom{x}{r+1} | t |^{x-r-1} e^{i\alpha(x-r-1)} \right| \\ &\quad \leqslant \frac{| x |}{r+1} \left| S_{n,r}^*(x - 1; t) - \binom{x-1}{r} | t |^{x-r-1} e^{i\alpha(x-r-1)} \right| \\ &\quad + \frac{2(n+1)}{r+1} \sum_{l=0}^{r} \left[\frac{1}{| t - 1 |^{l+1}} + \frac{| t | + \rho}{\rho^{l+1}} \right] \\ &\quad \times M_{r-l}(x - 1) | t |^{x-r+l-1} (n + 1)^{r-l} \gamma^n \\ &\quad \leqslant M_{r+1}(x)(n + 1)^{r+1} | t |^{x-r-1} \gamma^n, \end{aligned}$$

where

$$M_{r+1}(x) = \frac{x}{r+1} M_r(x-1)$$
$$+ \frac{2}{r+1} \sum_{l=0}^{r} \left[\frac{1}{|t-1|^{l+1}} + \frac{|t|+\rho}{\rho^{l+1}} \right] |t|^l M_{r-l}(x-1). \quad (6.11)$$

Note that $M_r(x)$ as defined by the recurrence (6.11) with $M_0(x) = M$ is positive and independent of n, as required. ∎

Let us now turn to the general case in which the zeros of $P(x)$ are not restricted to a single value. Let $r_1, r_2, ..., r_h$ be the distinct zeros of $P(x)$, and let r_j have multiplicity m_j. Then,

$$\sum_{j=1}^{h} m_j = k.$$

Let the data $\{y_\nu\}$ satisfy (1.2). Then, there are uniquely determined polynomials

$$q_j(x) \in \pi_{m_j - 1},$$

such that

$$y_\nu = \sum_{j=1}^{h} q_j(\nu)\, r_j^{\nu} \quad (6.12)$$

for all integers ν. We shall need the following lemma.

LEMMA 8. *Let $\{y_\nu\}$ satisfy* (1.2) *and therefore be of the form* (6.12). *Then, a cardinal spline $S(x)$ (of degree n) satisfies*

$$S(\nu) = y_\nu$$

for all ν if and only if it can be expressed in the form

$$S(x) = \sum_{j=1}^{h} S_{(j)}(x), \quad (6.13)$$

where, for each j, $S_{(j)}(x)$ is an exponential Euler spline of degree n and order $m_j - 1$, having the base r_j, and interpolating at the integers the data $\{q_j(\nu)\, r_j^{\nu}\}$.

Proof. The sufficiency is obvious. To prove the necessity, note that $1/P(x)$ has a condensed partial fraction expansion of the form

$$\frac{1}{P(x)} = \sum_{j=1}^{h} \frac{\alpha_j(x)}{(x - r_j)^{m_j}}, \quad (6.14)$$

where $\alpha_j(x)$ is a uniquely determined element of π_{m_j-1}. Now, let

$$\phi_j(x) = \frac{P(x)}{(x - r_j)^{m_j}} \qquad (j = 1, 2,..., h).$$

Then, multiplication of (6.14) by $P(x)$ gives

$$1 = \sum_{j=1}^{h} \alpha_j(x)\, \phi_j(x), \tag{6.15}$$

an identity in x. Now let $S_{(j)}(x)$ be defined by

$$S_{(j)}(x) = \alpha_j(E)\, \phi_j(E)\, S(x) \qquad (j = 1, 2,..., h). \tag{6.16}$$

Since $\alpha_j(x)$ and $\phi_j(x)$ are polynomials and a cardinal spline translated by an integer remains a cardinal spline of the same degree, $S_{(j)}(x)$ as defined by (6.16) is a cardinal spline of the same degree as $S(x)$. We must show that it has the required properties.

In view of (6.15), (6.13) is clearly satisfied. Moreover,

$$(E - r_j)^{m_j}\, S_j(x) = \alpha_j(E)\, P(E)\, S(x) = 0,$$

which shows that $S_{(j)}(x)$ is an exponential Euler spline of order $m_j - 1$ having the base r_j. It follows that $S_{(j)}(x)$ interpolates at the integers $\{\beta_j(\nu)\, r_j^{\nu}\}$, where $\beta_j(x)$ is some element of π_{m_j-1}. Then, (6.13) gives

$$\sum_{j=1}^{h} \beta_j(\nu)\, r_j^{\nu} = S(\nu) = y_\nu .$$

Since the representation (6.12) is unique, we must have

$$\beta_j(x) = q_j(x) \qquad (j = 1, 2,..., h),$$

and consequently,

$$S_{(j)}(\nu) = q_j(\nu)\, r_j^{\nu} \qquad (j = 1, 2,..., h) \tag{6.17}$$

for all integers ν. ∎

Note that if r_j and r_l are a conjugate complex pair, then it is easily seen from (6.14) and (6.16) that $S_{(j)}(x)$ and $S_{(l)}(x)$ are conjugates, so that $S(x)$ is a real-valued function, as it must be.

THEOREM 5. *Let $P(x)$ have no zero equal to* 0, 1 *or any negative quantity, and let y_ν be given by* (6.12). *Let $S_n(x)$ denote the unique cardinal spline of degree n satisfying* (1.1) *and interpolating $\{y_\nu\}$ at the integers. Then,*

$$\lim_{n\to\infty} S_n(x) = \sum_{j=1}^{h} q_j(x)\, |\, r_j\, |^{x} \exp(i\alpha_j x) \tag{6.18}$$

for all real x, where

$$r_j = |r_j| \exp(i\alpha_j), \qquad -\pi < \alpha_j < \pi, \qquad (j = 1, 2,..., h).$$

Proof. By Lemma 8, $S_n(x)$ can be expressed in the form (6.13), where (6.17) holds. There exist uniquely determined coefficients d_{jl} $(j = 1, 2,..., h;$ $l = 0, 1,..., m_j - 1)$ such that

$$\sum_{l=0}^{m_j-1} d_{jl} \binom{x}{l} = q_j(x) \qquad (j = 1, 2,..., h). \tag{6.19}$$

Consequently, by (6.17) and the definition of $S_{n,l}^*(x)$,

$$S_{(j)}(x) = \sum_{l=0}^{m_j-1} d_{jl} r_j^l S_{n,l}^*(x),$$

and therefore, by Theorem 4,

$$\lim_{n\to\infty} S_j(x) = \sum_{l=0}^{m_j-1} d_{jl} \binom{x}{l} |r_j|^x \exp(i\alpha_j x).$$

In view of (6.19), summing with respect to j then gives (6.18). ∎

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